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ASYMPTOTIC SOLUTIONS OF WAKES AND BOUNDARY LAYERS*

by

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1. *Introduction*

The asymptotic expansions of the solutions of boundary layer equations valid far downstream were critically examined by Stewartson [1]. It was pointed out that it was necessary to admit $\ln x$ terms in addition to terms involving only powers of x in order to insure the exponential decay of the solution with respect to y. The variable x is measured along the direction of the basic stream or along the boundary wall and y is the coordinate normal to x.

The development of higher order solutions for boundary layer along a flat plate [2, 3, 4, 5] was carried out as an asymptotic expansion of solutions of Navier Stokes equations for large Reynolds number, Re. In addition to the regular terms involving powers of Re, it was found necessary to add some terms involving $\ln$ Re.

In all those solutions, the logarithmic terms were added artificially after the breakdown of the regular solution. It is the purpose of this paper to show how to predict in advance whether the regular solution is going to break down or not and where the breakdown occurs if it does and to generate in a straightforward manner the complete asymptotic solution which will include the logarithmic terms if they should be present. This task will be accomplished by making use of the identification of asymptotic solution with perturbation solutions [6]. The perturbation parameter can be either the deviation from a given flow or the square root of the inverse of the Reynolds number and will be assigned for each problem. The perturbation equations are partial differential equations of two variables while in the analyses of predecessors the iteration or perturbation equations are ordinary differential equations in one variable with the dependence on the other, the x variable, preassigned.

The far wake problem of Goldstein [7] will be reexamined by the perturbation method in *2*. The main objectives of this paper can be achieved by following the classical approach [1, 7], using the boundary layer equation in Cartesian coordinates and linearizing the convective terms. However, in the present paper, the boundary layer equation in von Mises variables, the von Mises equation, [8] is used as the basic equation. The purpose for doing so is to show in *2.1* that for the far wake problem the von Mises equation is the equation to be used for linearization or iteration because the perturbation equations or the iteration equations preserve the condition of conservation of linear momentum. The leading term of the first perturbation solution of the von Mises equation yields the first term of the asymptotic solution for large x with the total momentum equal to that of initial profile or directly related to the drag of the body. There is no additional contribution to the first term of the asymptotic solution from

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the higher order perturbation solutions. In contrast, there will be additional contributions to the first term of the asymptotic solution from each successive perturbation or iteration of the boundary layer equation in Cartesian coordinates [1]. By starting from the von Mises equation, the perturbation equations have inhomogeneous parts different from those of Goldstein's analysis [7]. Nevertheless, the condition for the "breakdown" of the "regular" solution remains the same when the method of analysis of Goldstein is followed in *2.2*. When the perturbation equation is solved by the method of normal mode [9], the complete solution is obtained in a straightforward manner in *2.2* and the condition for the appearance of the logarithmic terms or that for the breakdown of the regular solution is clearly identified with the "resonance" condition in nonlinear vibration problems, i.e., there is a term in the inhomogeneous part, with the power of x equal to an eigenvalue and non-orthogonal to the corresponding eigenfunction. A general discussion of the appearance of the logarithmic terms in various powers in the higher order solutions is given in *2.3*.

The higher order solutions for the boundary layer along a flat plate are studied in *3.1*. The homogeneous part of the perturbation equation is identical to the linearized boundary layer equation for perturbations from the Blasius solution. The eigenvalues, eigenfunctions and the Green's function have been constructed in [10]. With the use of the Green's function, the higher order solutions are obtained in a straightforward manner including the logarithmic term. Its appearance can again be identified as the "resonance" condition, Three other examples of generation of the asymptotic solutions by means of the perturbation method are presented in *3.2*, *3.3*, and *3.4*.

2. *Two Dimensional Symmetric Wake*

Far downstream from the body, the pressure gradient becomes zero and the flow outside the wake is uniform with velocity U in the direction of the x-axis. The deviation of the velocity in the wake from the uniform velocity has been investigated by Tollmein [11], Goldstein [7] and Stewartson [1]. In their analyses, the boundary layer equations in x, y coordinates are used. For the first iterative solution, the governing equation is linearized by the Oseen type approximation

$$Uu_x - \nu u_{yy} = 0 \qquad (2.1)$$

where u is the x-component of velocity and ν is the kinematic viscosity. In the next iteration, the term $-(U-u)\,u_x - vu_y$ appears as the inhomogeneous term with the vertical component of the velocity, v, computed from the continuity equation. With the boundary conditions of $u_y(x,0) = 0$ and $u(x,\infty) \to U$ the integration of eq. (2.1) across the wake yields the conservation of the mass flux

$$\frac{d}{dx}\int_0^\infty (U-u)dy = 0 \qquad (2.2)$$

instead of the conservation of the momentum flux

$$\frac{d}{dx}\int_0^\infty u(U-u)dy = 0 \qquad (2.3)$$

The latter is obtained by the integration of the complete boundary layer equation. Stewartson [1] pointed out that the iteration solution will eventually fulfill eq. (2.3) as the number of iterations increase.

2.1 von Mises Equation

For the illustration of the perturbation techniques, the analysis can proceed with the same equation in x-y coordinates. However, in the present paper, the boundary layer equation in von Mises variables x, ψ will be used. The basic equation is [8]

$$u_x - \nu\left[uu_\psi\right]_\psi = 0 \tag{2.4}$$

where ψ is the stream function and is related to y by the equation

$$y = \int_0^\psi \frac{dx}{u(x,\psi)} . \tag{2.5}$$

The boundary conditions are

$$u \to U \quad \text{as} \quad \psi \to \infty, \quad x \geqq x_o \tag{2.6}$$

$$u_\psi = 0 \quad \text{at} \quad \psi = 0, \quad x \geqq x_o \tag{2.7}$$

and the initial condition at $x = x_o$ is

$$u(x_o, \psi) = U\left[1-\epsilon f(\psi)\right] , \quad \psi \geqq 0. \tag{2.8}$$

The small parameter ϵ is a measure of the deficiency in the initial profile, for example, $\epsilon = \left[U-u(x_o, 0)\right]/U$ and $f(0) = 1$. ϵ is assumed to be much less than unity but much larger than the inverse of the square root of the reference Reynolds number so that $u(x, \psi, \epsilon)$ obeys the boundary layer equation. The solution will be a function of $x-x_o$ and ψ and is invariant to the choice of the value for x_o. An optimum value for x_o will be defined later for the perturbation solution [6]. Consistent with the initial profile, the solution u is written in a power series of ϵ as follows:

$$u(x, \psi, \epsilon) = U\left[1-\epsilon u^{(1)}-\epsilon^2 u^{(2)}\ldots\right]$$

The perturbations equations are

$$u_x^{(1)} - \nu U\, u_{\psi\psi}^{(1)} = 0 \tag{2.9a}$$

$$u_x^{(2)} - \nu U\, u_{\psi\psi}^{(2)} = -\frac{\nu U}{2}\left\{\left[u^{(1)}\right]^2\right\}_{\psi\psi} \tag{2.9b}$$

$$u_x^{(n)} - \nu U\, u_{\psi\psi}^{(n)} = \frac{-\nu U}{2}\left\{\sum_{j=1}^{n-1} u^{(j)}\, u^{(n-j)}\right\}_{\psi\psi}, \quad n = 2, 3, \ldots \tag{2.9c}$$

The boundary conditions for $n = 1, 2, 3, \ldots$ are

$$u^{(n)} \to 0 \text{ as } \psi \to \infty \quad x \geq x_o \tag{2.10}$$

and

$$u_\psi^{(n)} = 0 \text{ at } \psi = 0 \quad x \geq x_o \tag{2.11}$$

The initial conditions at $x = x_o$ are

$$u^{(1)} = f(\psi) \tag{2.12}$$

and

$$u^{(n)} = 0 \qquad n = 2,3,4.. \tag{2.13}$$

It is clear that by means of von Mises variables, the continuity equation is uncoupled from the determination of $u^{(n)}(x,\psi)$. An additional and essential advantage can be found by the integration of the perturbation equations across the wake and by applying the boundary conditions. The result is

$$\frac{d}{dx}\int_0^\infty u^{(n)}\,d\psi = \frac{d}{dx}\int_0^\infty u^{(n)}u\,dy = 0 \qquad n = 1,2,3.. \tag{2.14}$$

From the initial conditions for $u^{(1)}$, eq. (2.14) becomes

$$\int_0^\infty u^{(1)}d\psi = \int_0^\infty f(\psi)d\psi$$

or

$$\int_0^\infty\left\{\left[U-\epsilon u^{(1)}\right]u\right\}_{x>x_0} dy = \int_0^\infty\left\{\left[U-\epsilon u^{(1)}\right]u\right\}_{x=x_0} dy \tag{2.15}$$

For $n = 2,3,4..$, eq.(2.13) and eq.(2.14) yield

$$\int_0^\infty u^{(n)}d\psi = \int_0^\infty u^{(n)}u\,dy = 0 \tag{2.16}$$

Equation (2.15) and (2.16) state the facts that the first perturbation solution preserves the momentum flux of the initial profile and that there is no contribution to the momentum flux from the higher order solutions.

Equation (2.9a) or the homogeneous equation of eq. (2.9c) is identical to a linear heat conduction equation. When the initial profile $f(\psi)$ decays exponentially with respect to ψ, the solution for $x>x_0$ possesses the same property and the boundary condition of eq. (2.10) is replaced by the stronger condition

$$u^{(n)} \to 0\ (\psi^{-\alpha}) \text{ for any } \alpha > 0 \text{ as } \psi \to \infty. \tag{2.17}$$

2.2 Perturbation Solutions

With the condition of exponential decay as $\psi \to \infty$ and the condition of symmetry with respect to ψ, the eigenvalues, λ, of the homogeneous equation of eq. (2.9) are odd integers, i.e., $\lambda = 1,3,5..$ and the eigenfunctions are $x^{-\lambda/2}\Phi_\lambda(\xi)$ where $\xi = \psi(4\nu Ux)^{-\frac{1}{2}}$ and $\Phi_\lambda(\xi)$ is the λ-th derivative of erf (ξ) [12, 13]. The solution for $u^{(1)}$ can be expressed as a sum of the eigenfunctions, [6],

$$u^{(1)}(x,\psi) = \Sigma A_\lambda \Phi_\lambda(\xi)(x_0/x)^{\lambda/2} \quad ,\ \lambda = 1,3,5... \tag{2.18}$$

$$\text{where } A_\lambda = \frac{\int_0^\infty f\left[\xi_0(4\nu Ux_0)^{\frac{1}{2}}\right](\exp\xi_0^2)\Phi_\lambda(\xi_0)d\xi_0}{\int_0^\infty \exp\xi_0^2\ \Phi_\lambda^2(\xi_0)d\xi_0} \tag{2.19}$$

In particular, A_1 becomes:

$$A_1 = \int_0^\infty f(\psi)d\psi/(4\nu Ux_0)^{\frac{1}{2}} \tag{2.20}$$

When A_3 is set equal to zero, an optimum value for x_o is obtained, [6]

$$x_o = (2\nu U)^{-1} \int_o^\infty \psi^2 f(\psi)\, d\psi / \int_o^\infty f(\psi) d\psi > 0 \tag{2.21}$$

and eq. (2.18) becomes

$$u^{(1)}(x,\psi) = \left[\int_o^\infty f(\psi)d\psi\right]\left[(4\nu Ux)^{-\frac{1}{2}} \Phi_1(\xi)\right] + o(x^{-5/2}) \tag{2.22}$$

The first term of the perturbation velocity $-\epsilon U u^{(1)}$ represents a simple source with its strength equal to $\int_o^\infty [U-u]u\, dy$ at the initial station [12].

For $u^{(2)}, u^{(3)}$, etc. the boundary conditions and the initial conditions are homogeneous, they are nontrivial because of the inhomogeneous terms in their differential equations. The solutions can be expressed in terms of integrals and the integrals are expanded in asymptotic series for large x as done by Stewartson. In order to show the condition for the occurrence of the ℓn terms, it is clearer to use the method of separation of variables and the method of normal modes [9].

Since $u^{(1)}$ is expressed in terms of the variable x and ξ, eq. (2.9b) can be written in terms of the new variables as follows:

$$L\, u^{(2)} - 4x\, u^{(2)}_x = \frac{1}{2}\left\{\left[u^{(1)}\right]^2\right\}_{\xi\xi}$$

$$= \sum_{n=2,4,6..} F_n(\xi)(x/x_o)^{-n/2} \tag{2.23}$$

where
$$L = \frac{d^2}{d\xi^2} + 2\xi \frac{d}{d\xi} \tag{2.23a}$$

and
$$F_n(\xi) = \frac{1}{2} \sum_{k=1,3,5..}^{n-1} A_k A_{n-k} (\Phi_k \Phi_{n-k})_{\xi\xi} \tag{2.23b}$$

for each value of n, the particular integer can be written as $x^{-n/2} H_n(\xi)$ where $H_n(\xi)$ is the solution of the equation,

$$H_n''(\xi) + 2\xi H_n'(\xi) + 2n\, H_n(\xi) = F_n(\xi), \quad n = 2,4,6,.. \tag{2.24}$$

subjected to the boundary conditions of $H_n'(0) = 0$ and $H_n(\infty) \to 0$ exponentially. Since n is even and cannot be one of the eigenvalues for the homogeneous equation which are odd integers, the method of separation of variables works and $H_n(\xi)$ can be obtained by making use of $\Phi_n(\xi)$, one of the solutions to the homogeneous equation. The result is

$$H_n(\xi) = \Phi_n(\xi) \int^\xi \Phi_n^{-2}(\bar\xi)\, \exp(-\bar\xi^2) d\bar\xi \int_\infty^{\bar\xi} F_n(\xi')\Phi_n(\xi')\exp(\xi')^2 d\xi'$$

$$+ C_n\, \Phi(\xi) \tag{2.25}$$

where C_n is to be determined by the condition $H_n'(0) = 0$.
The second perturbation solution is,

$$u^{(2)}(x,\xi) = \sum_{n=2,4,6}^{\infty} (x_o/x)^{n/2} H_n(\xi) + \sum_{\lambda=3,5..}^{\infty} B_\lambda \Phi_\lambda(\xi)(x_o/x)^{\lambda/2}$$

$$= \frac{1}{2} A_1^2\left[\Phi_1^2(\xi) + \Phi_2(\xi)\, \mathrm{erf}(\xi)\right](x_o/x) + 0(x^{-3/2}) \tag{2.26}$$

where B_λ s' are determined by the initial condition at x_o. The result is

$$B_\lambda = - \int_0^\infty \sum_{n=2,4,6}^{\infty} H_n(\xi)\Phi_\lambda(\xi)\exp \xi^2 d\xi \Big/ \int_0^\infty \Phi_\lambda^2(\xi) \exp \xi^2 d\xi \qquad (2.26a)$$

The coefficient B_1 is set equal to zero due to the fact pointed out before that the perturbation equation conserves the momentum integral. It is hard to see $B_1 = 0$ directly from the equation for B_λ unless $H_n(\xi)$ were expanded in series of the eigenfunctions Φ_λ. Since this expansion will be carried out for the next order approximate solution, it will not be repeated here.

For the determination of the next order solution, $u^{(3)}$, eq. (2.9c) is rewritten in terms of variables x and ξ,

$$Lu^{(3)} - 4xu_x^{(3)} = \left[u^{(1)}u^{(2)}\right]_{\xi\xi} = \sum_{j=3,4,5}^{\infty} (x_o/x)^{j/2} G_j(\xi) \qquad (2.27)$$

where $$G_j(\xi) = \sum_{k=1,3,5}^{j=2} A_k \left[\Phi_k(\xi)H_{j-k}(\xi)\right]_{\xi\xi} \quad \text{for } j = 3, 5, 7\ldots \qquad (2.27a)$$

and $$G_j(\xi) = \sum_{k=1,3,5}^{j=1} A_k B_{j-k} \left[\Phi_k(\xi)\Phi_{j-k}(\xi)\right]_{\xi\xi} \quad \text{for } j = 4, 6, 8 \qquad (2.27b)$$

If $u^{(3)}$ is also assumed to be the sum of products, $X^{-j/2} K_j(\xi)$, $K_j(\xi)$ has to be the solution of the same equation for H_j, i.e. eq. (2.24) with F_j replaced by G_j,

$$LK_j + 2jK_j = G_j \quad j = 3, 4, 5\ldots \qquad (2.28)$$

subjected to the boundary conditions, $K'_j(0) = 0$ and $K_j \to 0$ exponentially as $\xi \to \infty$.

When j is an even integer, the solution K_j can be obtained in an identical manner as H_j given by eq. (2.25).

When j is an odd integer, which is an eigenvalue of the differential equation, the solution K_j does not exist if G_j is not orthogonal to the eigenfunction of Φ_j. The condition is

$$\left[G_j, \Phi_j\right] = \int_0^\infty G_j\Phi_j \exp \xi^2 d\xi \neq 0 \qquad (2.29)$$

In particular, for the first odd integer of j's, j=3, eq. (2.29) reduces to

$$\left[G_3, \Phi_3\right] = \int_0^\infty G_3\Phi_3 \exp \xi^2 d\xi \neq 0$$

or

$$\int_0^\infty G_3\left[2\xi^2 - 1\right] d\xi \neq 0 \qquad (2.30)$$

This is exactly Goldstein's condition [7] for the breakdown of his third iteration solution when the variable ξ is related to Goldstein's variable η by the equation $2\xi^2 = \eta^2$.

Once the cause for the breakdown is traced to the existence of inhomogeneous terms which are proportional to the eigenfunctions $x^{-\lambda/2}\Phi_\lambda(\xi)$ of the partial differential equation eq. (2.27), it is evident that these terms should be separated from the rest. The inhomogeneous term $x^{-j/2}G_j(\xi)$ with j equal to an odd integer can be split into two parts, the first part is proportional to the eigenfunction and the second part is orthogonal to it. The particular integral for the second part can be obtained in the same manner as for H_j. The particular integral for the first part can be obtained by the method of normal modes. Instead of obtaining these two parts separately, both of them and the part with j's equal to even integers will be obtained together by the method of normal modes [9].

The inhomogeneous terms will be rewritten in series of the eigenfunctions $\Phi_\lambda(\xi)$, i.e.,

$$\sum_{j=3,4,5}^{\infty} (x/x_0)^{-j/2} G_j(\xi) = \sum_{\lambda=3,5}^{\infty} \Phi_\lambda(\xi) \sum_{j=3,4,5}^{\infty} c_{\lambda j}(x/x_0)^{-j/2} \tag{2.31}$$

where

$$c_{\lambda j} = \int_0^\infty G_j(\xi)\Phi_\lambda(\xi) \exp \xi^2 \, d\xi \Big/ \int_0^\infty \Phi_\lambda^2 \exp \xi^2 \, d\xi \tag{2.32}$$

The solution $u^{(3)}$ will be expressed by the following series,

$$u^{(3)} = \sum_{\lambda=1,3,5..}^{\infty} X_\lambda(x) \, \Phi_\lambda(\xi).$$

The partial differential equation, eq. (2.27) yields the following equation for $X_\lambda(x)$,

$$2x \, X_\lambda'(x) + \lambda X_\lambda(x) = -\tfrac{1}{2} \sum_{j=3,4,5}^{\infty} c_{\lambda j}(x_0/x)^{j/2} \quad \text{for } \lambda = 3, 5, 7, \ldots$$

The solution of this equation subjected to the homogeneous initial condition at $x = x_0 > 0$ is

$$X_\lambda = \tfrac{1}{2} \sum_{\substack{j=3,4,5 \\ j\neq\lambda}}^{\infty} \frac{c_{\lambda j}}{j-\lambda} \left[(x_0/x)^{j/2} - \left(\frac{x_0}{x}\right)^{\lambda/2} \right] - \frac{c_{\lambda\lambda}}{4} (x_0/x)^{\lambda/2} \ln(x/x_0)$$

The third perturbation solution is

$$\begin{aligned} u^{(3)} = & - \ln(x/x_0) \sum_{\lambda=3,5,7}^{\infty} \frac{c_{\lambda\lambda}}{4} (x_0/x)^{\lambda/2} \Phi_\lambda(\xi) \\ & + \tfrac{1}{2} \sum_{\lambda=3,5,7}^{\infty} \Phi_\lambda(\xi) \sum_{\substack{j=3,4,5 \\ j\neq\lambda}}^{\infty} \frac{c_{\lambda j}}{j-\lambda} \left[\left(\frac{x_0}{x}\right)^{j/2} - \left(\frac{x_0}{x}\right)^{\lambda/2} \right] \end{aligned} \tag{2.33}$$

The first series contains the extra factor $\ln(x/x_0)$. In particular for $\lambda = 3$, c_{33} given by eq. (2.32) is equal to $A_1^3/(\pi 3^{\frac{1}{2}})$ and $u^{(3)}$ can be written as

$$u^{(3)} = -\frac{A_1^3}{4\pi 3^{\frac{1}{2}}} \left(\frac{x_0}{x}\right)^{3/2} \ln x \; \Phi_3(\xi) + 0\,(x^{-3/2})$$

This equation together with eq. (2.22) for $u^{(1)}$ and (2.26) for $u^{(2)}$ yield the solution for the x-component of velocity in the wake region $x \geq x_0$,

$$\begin{aligned} u = U \Big\{ & 1 - \epsilon A_1 (x_0/x)^{\frac{1}{2}} \Phi_1(\xi) + \\ & -\tfrac{1}{2}\epsilon^2 A_1^2 (x_0/x) \left[\Phi_1^2(\xi) + \Phi_2(\xi) \operatorname{erf}(\xi) \right] + \\ & + \frac{\epsilon^3 A_1^3}{4\pi 3^{\frac{1}{2}}} (x_0/x)^{3/2} \ln x \; \Phi_3(\xi) \\ & + 0(\epsilon x^{-5/2}) + 0(\epsilon^2 x^{-3/2}) \Big\} \end{aligned}$$

From the dependence of the inhomogeneous terms on x, it can be concluded that the contributions from the higher order perturbation $\epsilon^4 u^{(4)}$ are at most of the order of $\epsilon^4 x^{-3/2}$ or $\epsilon^4 x^{-5/2} \ln x$ and of higher order than $\epsilon^2 x^{-3/2} \ln x$

either as $\epsilon \rightarrow 0$ or $x \rightarrow \infty$. The last statement is valid for any term $\epsilon^n u^{(n)}$ with $n > 3$.

Along the line of symmetry, $\psi = 0$ and $\xi = 0$, the asymptotic expression in x for u is

$$u = U\left\{1 - Ax^{-\frac{1}{2}} - \tfrac{1}{2}A^2 x^{-1} - (1/8)3^{-\frac{1}{2}} A^3 x^{-3/2} \ln x + 0(x^{-3/2})\right\}$$

where

$$A = 2\epsilon A_1 (x_0/\pi)^{\frac{1}{2}} = \int_0^\infty \epsilon f(\psi)\, d\psi/(\pi \nu U)^{\frac{1}{2}}$$

$$= \int_0^\infty (U-u)u\, dy/(\pi\nu U)^{\frac{1}{2}} = \tfrac{1}{2}D/(\rho^2 \pi\nu U)^{\frac{1}{2}}$$

The constant A which is proportional to the integral of the initial profile or to the momentum thickness is identical with the one used by Goldstein [7] who relates A to the drag, D, of the body when the initial profile in the wake is unspecified. The asymptotic expression including the $x^{-3/2} \ln x$ term is in agreement with that by Crane [14] who corrected a slip in Stewartson's analysis. Crane [14] applied Lighthill's technique [15] to remove the logarithmic term by transforming x to a new variable. However, his solution in terms of the physical variable x still contains the logarithmic term. In the spirit of Lighthill's technique [15], the undesirable singularity is usually allowed in the plane of the transformed variables but removed in the plane of the physical variables. In the present problem, the first logarithmic term appears as $\epsilon^3 x^{-3} \ln x/x_0$. It is uniformly smaller than 1, ϵx^{-1} and $\epsilon^2 x^{-2}$ for $x \geq x_0$ and therefore does not invalidate the perturbation scheme.

2.3 Logarithmic Terms in Higher Order Solutions

Once the reason for the appearance of the logarithmatic term is understood, it becomes relatively easy to keep track of the higher order solution. In the third order terms, the ϵ^3 terms, logarithmic terms are of the type $x^{-\lambda/2} \ln x\, \Phi_\lambda(\xi)$. In the ϵ^4 terms there are terms of the type $x^{-n/2} \ln x\, \Phi_\lambda(\xi)$ where n is any integer due to the product of $u^{(1)}$ and $u^{(3)}$ and the square of $u^{(2)}$ in the inhomogeneous part. In the ϵ^5 terms there are in addition to these typical $\ln x_2$ terms but also terms of the type $x^{-\lambda/2}(\ln x)^2\, \Phi_\lambda(\xi)$. The terms with $(\ln x)^2$ are due to the products of $u^{(2)}$ and $u^{(3)}$ and of $u^{(1)}$ and $u^{(4)}$ which produce inhomogeneous terms of the type $x^{-\lambda/2} \ln x\, \Phi_\lambda(\xi)$. The leading $(\ln x)^2$ is proportional to $\epsilon^5 A^5 x^{-5/2} (\ln x)^2 \Phi_5(\xi)$. It can be generalized to the statement that the leading $(\ln x)^m$ term is proportional to $(\epsilon A)^{2m+1} x^{-(2m+1)/2} (\ln x)^m \Phi_{2m+1}(\xi)$.

3. Perturbations from Similar Solutions in Boundary Layer

For a given similar solution, say the Blasius solution, the homogeneous part of the perturbation equation will be the same, while the inhomogeneous part will be different for different perturbation parameters. In the study of higher order solutions for boundary layer along a flat plate in *3*, the expansion parameter ϵ is the inverse of the square root of the reference Reynolds number, the governing equations are the Navier Stokes equations, and the inhomogeneous terms include those omitted in the boundary layer equation and the interaction with the flow field outside the boundary layer. When the perturbation is due to small deviation of pressure from the constant value *3.2* or due to the small deviation of the initial profile from the similar profiel, *3.3*, the small perturbation parameter is usually much larger than the inverse of the square root of the Reynolds number and, therefore, the governing equation is the boundary layer equation. Since the homogeneous parts of the perturbation equations are the same and the boundary conditions can be reduced to the same homogeneous conditions, the eigenvalues, eigenfunctions and the Green's functions are the same. Once the different inhomogeneous parts are obtained, the same treatment can be employed. A general discussion of the appearance of logarithmic terms and powers of logarithmic terms is given at the end of *3.1* for perturbations from the Blasius solutions. *3.4* gives a general discussion for perturbations from a similar solution with pressure gradient.

3.1 Higher Order Solutions for Boundary Layer over a Flat Plate

The higher order solution for boundary layer over a semi-infinite plate was obtained by Alden using parabolic coordinates [16]. The solution has the defect that the resulting vorticity decays only algebraically for large y or η. To enforce the condition of exponential decay, Goldstein corrected Alden's solution by the addition of a term involving $\ell n\ x$ or $\ell n\ Re_x$ where Re_x is the local Reynolds number. In this section, the perturbation method will be used to generate the asymptotic solution including the logarithmic term whose appearance will again be identified as the "resonance" phenomenon of an inhomogeneous term with one of the eigensolutions.

For the sake of some variation from the classical method of approach with parabolic coordinates, the Cartesian coordinates x, y will be used in the analysis. Near the plate represented by the positive x-axis and at sufficient distance from the leading edge (say $x \geq x_0$), the leading term of the solution obeys the boundary layer equation. It is obtained systematically by a stretching of the y-coordinate to Y with $Y = y/\epsilon$. The small parameter ϵ represents the inverse of the square root of the Reynolds number, Re, with respect to a reference length L. The stream function for the boundary layer region will be represented by a Taylor series in ϵ, i.e.,

$$\psi(x,Y,\epsilon) = \epsilon\psi_0(x,Y) + \epsilon^2\psi_1(x,Y) + \epsilon^3\psi_2(x,Y) + \dots \tag{3.1}$$

The leading term is given by the Blasius solution,

$$\epsilon\psi_0(x,Y) = (2\nu Ux)^{\frac{1}{2}} f_0(\eta) = \epsilon U(2xL)^{\frac{1}{2}} f_0(\eta) \tag{3.2}$$

where f_0 is the Blasius function [8] and η is equal to $y(2\nu x/U)^{-\frac{1}{2}}$ or $Y/(2xL)^{\frac{1}{2}}$. For large η, f_0 behaves as $\eta - \beta + 0[\exp(-\eta^2)]$ with $\beta = 1.217$.

The stream function $\bar{\psi}(x,y,\epsilon)$ for the outer region will also be represented as a Taylor series in ϵ as follows [5]:

$$\bar{\psi}(x,y,\epsilon) = Uy - U\epsilon\beta\left\{[r+x]L\right\}^{\frac{1}{2}} + \epsilon^2\bar{\psi}_2(x,y) + \epsilon^3\bar{\psi}_3(x,y) + \dots \tag{3.3}$$

The first term represents the uniform flow and the second term the disturbance due to displacement thickness of a Blasius profile [17]. Since the induced pressure and tangential velocity due to the second term vanish at $y = 0$, the second term $\psi_1(x,Y)$ in the boundary layer vanishes and consequently the next order outer solution $\bar{\psi}_2(x,y)$ also vanishes, i.e.,

$$\psi_1(x,Y) = 0 \quad \text{and} \quad \bar{\psi}_2(x,y) = 0 \tag{3.4}$$

The implication of eqs. (3.2) and (3.4) on the initial data at $x = x_0$ will be discussed at the end of this section.

Due to the dependence of ψ_1 on x and η, the variables x, Y will be replaced by x, η. After the substitution of eqs. (3.1) and (3.2) into the normal component of the Navier Stokes equation, the coefficients of ϵ yield

$$p_{2,\eta}/(\rho U^2) = \left[L/(2x)\right]\left[\eta f_o f_o' - f_o^2 + f_o''\right]_\eta \tag{3.5}$$

The condition of matching with the outer solution [5, 18, 19] gives

$$\lim_{\eta\to\infty}\left[p_2\ (x,\eta) - \eta(2xL)^{\frac{1}{2}}\ \bar{p}_{1,y}(x,0)\right] \longrightarrow \bar{p}_2\ (x,0) = -\rho U^2\beta^2 L/(4x)$$

or

$$\lim_{\eta\to\infty}\left[p_2\ (x,\eta) - \rho U^2\eta\beta L/(2x)\right] \longrightarrow -\rho U^2\beta^2 L/(4x)$$

and the integration of eq. (3.5) with respect to η gives

$$p_2/(\rho U^2) = \left[L/(2x)\right]\left[\eta f_o f_o' - f_o^2 + \eta f_o'' + \beta^2/2\right] \tag{3.6}$$

From the condition for matching of u_2 with the outer solution, the behavior of u_2 as $\eta\to\infty$ is $u_2 + U\beta Y L^{\frac{1}{2}}/(2x)^{3/2} \longrightarrow 0$. In the change of the dependent variables, the stream function $\psi_2(x,\eta)$ will be related to the new function $f_2(x,\eta)$ as follows

$$\psi_2(x,Y) = U(2xL)^{\frac{1}{2}}\ f_2(x,\eta) - \tfrac{1}{2}U\beta Y^2 L^{\frac{1}{2}}/(2x)^{3/2} \tag{3.7}$$

so that

$$u_2(x,Y) + U\beta Y L^{\frac{1}{2}}/(2x)^{3/2} = \epsilon\psi_{2,y} + U\beta Y L^{\frac{1}{2}}/(2x)^{3/2} = U f_{2,\eta}(x,\eta)$$

The boundary conditions for f_2 will become homogeneous,

$$f_2(x,o) = f_{2,\eta}\ (x,0) = 0 \tag{3.8}$$

and $\quad f_{2,\eta}\ (x,\eta\to\infty) \longrightarrow 0.$

The last condition will be replaced by a stronger condition,

$$f_{2,\eta}(x,\eta\to\infty) \longrightarrow 0 \qquad \text{exponentially,} \tag{3.9}$$

in order to fulfill the condition of exponential decay of vorticity.

After the substitution of eqs. (3.1), (3.2), (3.6) and (3.7) into the x-component of the Navier Stokes equation, the coefficients of ϵ^2 yield

$$M\ f_2(x,\eta) = H(\eta)\ L/(2x) \tag{3.10}$$

where

$$M = \frac{\partial^3}{\partial\eta^3} + f_o\frac{\partial^2}{\partial\eta^2} + f_o'' - 2xf_o'\frac{\partial^2}{\partial x\,\partial\eta} + 2xf_o''\frac{\partial}{\partial x} \tag{3.10a}$$

and

$$H(\eta) = -\beta^2 - \eta^2 f_o'^2 - 6\eta f_o'' - \eta f_o f_o' + \eta^2 f_o f_o'' + 2f_o^2 + \beta\left[f_o - \tfrac{1}{2}\eta^2 f_o'' + 2\eta f_o'\right]. \tag{3.10b}$$

It can be verified easily that $H(\eta)$ vanishes exponentially as $\eta \to \infty$. The homogeneous part of eq. (3.15) is identical with the linearized boundary layer equation when perturbed from Blasius solution [10]. The eigen solutions of the homogeneous equation subjected to the homogeneous boundary conditions of eqs. (3.8) and (3.9) have the form $(x/x_o)^{-\lambda_k/2} N_k(\eta)$ where the eigenvalues λ_k, are 2, 3.774, 5.629 ..... . The first twenty eigenvalues are tabulated in [20]. Only the first eigenvalue is an integer and the first eigenfunction is proportional to the x-derivative of the Blasius solution, i.e., $N_1(\eta) = [\eta f_o' - f_o]/f_o''(0)$ where $f_o''(0) = 0.4696$. Since the power of x in the inhomogeneous term is equal to $-\lambda_1/2$ and $H(\eta)$ is in general not orthogonal to $N_1(\eta)$, a logarithmic term is expected. Although the method of normal mode can be used here again, the equivalent method of Green's function will be used since the Green's function is available in [10]. The particular integral $f_{2,p}$ is

$$f_{2,p} = - \sum_{k=1}^{\infty} \left\{ \frac{N_k(\eta)}{C_k} \int_0^{\infty} d\bar{\eta}\, H(\bar{\eta}) \left[\frac{f_o'^2}{f_o''} \left(\frac{N_k}{f_o'} \right)' \right]_{\eta=\bar{\eta}} \right\} \int_{x_o}^{x} \frac{d\bar{x}}{2x} \left(\frac{x}{\bar{x}} \right)^{-\lambda_k/2} \left(\frac{L}{2\bar{x}} \right) \qquad (3.11)$$

where $C_k = \int_0^{\infty} (f_o'^4/f_o') \left[(N_k/f_o')' \right]^2 d\eta$ is the normalizing constant. The term inside the curly brackets represents the resolution of $H(\eta)$ to the k^{th} normal mode. The integral with respect to $\bar{x}$ can be carried out and the definition of N_1 can be introduced to simplify the first term. The result is

$$f_{2,p} = - \frac{L}{2x} \ln \frac{x}{xo} (\eta f_o' - f_o) \int_0^{\infty} f_o(\bar{\eta}) H(\bar{\eta}) d\bar{\eta}$$
$$+ \sum_{k=2,3,4}^{\infty} \frac{B_k}{(2-\lambda_k) C_k} \left(\frac{L}{2x} \right) \left[1 - \left(\frac{x_o}{x} \right)^{\lambda_k/2 - 1} \right] N_k(\eta)$$

where

$$B_k = \int_0^{\infty} d\bar{\eta} H(\bar{\eta}) \left[(f_o'^2/f_o'') \quad (N_k'/f_o')' \right]_{\eta=\bar{\eta}}$$

The integral in the first term is found to be -1.658. Together with the complimentary integral, the solution is

$$f_2 = 0.829\ (L/x) \ln Re_x\ (\eta f_o' - f_o)$$
$$+ \tfrac{1}{2}(L/x) \sum_{k=2,3...}^{\infty} B_k N_k(\eta) / \left[(2-\lambda_k) C_k \right] \qquad (3.12)$$
$$+ \sum_{k=1,2}^{\infty} A_k N_k(\eta)\ (x/x_o)^{-\lambda_k/2}$$

Where Re_x denotes the local Reynolds number. The coefficients A_k are undefined because the initial profile at the station $x = x_o$ is not yet available. In replacing $\ln x/x_o$ in the first term by $\ln Re_x$, the difference $\ln Re_{xo}$ is absorbed in the unknown constant A_1. A_k's will be independent of ϵ or the Reynolds number if Re_{xo} is of the order of unity and the initial profile at the station $x = x_o$ differs from the Blasius solution by an order of ϵ^2. Indeed Imai [3] patched at $Re_{xo} = 1$ with the result of Carrier and Lin [21] for viscous flow near leading edge and obtained a finite value for A_1. The validity of this patching is still in doubt [4].

Although the inhomogeneous term in eq. (3.10) is different from the corresponding term when parabolic coordinates are used, the integral of the product of f_o and the inhomogeneous term with respect to η from 0 to ∞

is independent of the coordinate system. From eqs. (3.1), (3.7) and (3.12), the coefficient of skin friction is given by $C_f \sim 0.664(Re_x)^{-\frac{1}{2}} + 0.551(Re_x)^{-3/2} \ell n Re_x + 0(Re_x)^{-3/2}$. It is in agreement with the results of Goldstein and Imai.

It should be pointed out that the eigenvalues λ_k with $k \geq 2$ have the following properties: i) they are not integers, ii) they do not differ from each other by an integer, iii) the sum of any number of eigenvalues is not an eigenvalue. Since the inhomogeneous parts of higher order equations are formed by the products of perturbation solutions with terms involving powers of x equal to half of a negative integer or a half negative integer minus half the sum of several eigenvalues, the "resonance" phenomenon can occur only with the first eigenvalue $\lambda_1 = 2$. The conditions for the appearance of a "resonance" term or a $x^{-1} \ell n\, x$ term in the solutions are: i) there is a term in the inhomogeneous part which is proportional to x^{-1} i.e. of the form $x^{-1} G(\eta)$ and ii) the function $G(\eta)$ is not orthogonal to the first eigenfunctions

$$N_1(\eta), \quad \text{i.e.} \quad \int_0^\infty f_o(\eta)\, G(\eta) d\eta \neq 0.$$

Since the negative power of x in the inhomogeneous term for the third or higher order perturbations are greater than λ_1, resonance phenomenon will not occur. The ℓnx terms will appear only through the products with the $\epsilon^2 x^{-1} \ell n\, x$ in $\epsilon^2 f_2(x,\eta)$. For example, it is expected to have $\epsilon^3 x^{-3/2} \ell n\, x$ terms in $\epsilon^3 f_3$, $\epsilon^4 x^{-2} \ell n\, x$ and $\epsilon^4 x^{-1-\lambda_k/2}\ \ell n\, x$ terms in $\epsilon^4 f_4$ and the leading $(\ell n\, x)^2$ term will appear as $\epsilon^5 x^{-5/2} (\ell n\, x)^2$ in $\epsilon^5 f_5(x,\eta)$.

3.2 Boundary Layer with Small Pressure Gradient

When the parameter ϵ is used as a measure of the small variation of pressure along the surface after the station $x = x_o > 0$, the pressure gradient can be written as

$$p_x/(\rho U^2) = \epsilon \bar{p}_x(x) \qquad x \geq x_o$$

The velocity outside the boundary layer will differ from U by an order of ϵ and will be consistent with $\bar{p}$. The small parameter ϵ is assumed to be much larger than the inverse of the square root of the reference Reynolds number, so that the governing equation is the boundary layer equation. The stream function ψ can be written as

$$\psi(x, y, \epsilon) = (2\nu Ux)^{\frac{1}{2}} \left\{ f_o(\eta) + \epsilon f_1(x,\eta) + \epsilon^2 f_2(x,\eta) + \dots \right\} \tag{3.13}$$

where f_o is the Blasius function and $\eta = y/(2x\nu/U)^{\frac{1}{2}}$. The functions f_1 and f_2 are governed by the equations

$$Mf_1 = 2x\bar{p}_x \tag{3.14a}$$

and

$$Mf_2 = 2x(f_{1,\eta} f_{1,\eta x} - f_{1,x} f_{1,\eta\eta}) - f_1 f_{1,\eta\eta} \tag{3.14b}$$

The linear operator M is defined in eq. (3.10a). To be specific, the pressure variation assumes the following form

$$(p-P)/(\rho U^2) = \epsilon \sum_{n=1,2,3\dots} a_n (x/x_o)^{-n/2} \tag{3.15}$$

and the pressure gradient becomes

$$\bar{p}_x = -(2x)^{-1} \sum_{n=1,2,3} n a_n (x/x_o)^{-n/2}$$

Since f_1' and the inhomogeneous part in eq. (3.14a) do not vanish as $\eta \to \infty$, a new function $F_1(x,\eta)$ is introduced, with

$$F_1(x,\eta) = f_1(x,\eta) + f_o \sum_{n=1,2,3\ldots} a_n (x/x_o)^{-n/2} \tag{3.16}$$

such that

$$MF_1 = -\sum_{n=1,2\ldots} a_n (x/x_o)^{-n/2} \left[n(1-f_o'^2) - (1-n) f_o f_o'' \right] \tag{3.17}$$

F_1 fulfills the homogeneous boundary conditions, $F_1 = F_{1,\eta} = 0$ at $\eta = 0$ and $F_{1,\eta} \to 0$ exponentially as $\eta \to \infty$. The eigenvalues λ_k and eigenfunction $N_k(\eta)$ for $MF_1 = 0$, are the same as those in the preceeding example.

A ℓnx term is expected only from the inhomogeneous term with $n = 2 = \lambda_1$. The solution for $F_1(x,\eta)$ can be obtained from the Green's function [10] and the result is

$$F_1(x,\eta) = -a_1 (x/x_o)^{-\frac{1}{2}} \left[\int_0^\eta d\bar{\eta}\, f_o''(\bar{\eta}) \int_0^{\bar{\eta}} \frac{\eta' - \beta}{f_o''(\eta')} d\eta' - f_o(\eta) \right]$$

$$- a_2 (x/x_o)^{-1} \ell n(x/x_o)\, (f_o - \eta f_o') \int_0^\infty 2 f_o (1 - f_o'^2 + \tfrac{1}{2} f_o f_o'') d\eta$$

$$+ 0(x^{-1})$$

The remainder of the particular integer and the contributions from the initial conditions are represented by the symbol $0(x^{-1})$. The solution for $f_1(x,\eta)$ is

$$f_1(x,\eta) = -a_1 (x/x_o)^{-\frac{1}{2}} \int_0^\eta d\bar{\eta}\, f_o''(\bar{\eta}) \quad \frac{\eta' - \beta}{f_o''(\eta')} d\eta'$$

$$+ 1.70\, a_2 (x/x_o)^{-1}\, \ell n(x/x_o)\, (\eta f_o' - f_o) + 0(x^{-1}) \tag{3.18}$$

Eq. (3.14b) can now be written as

$$Mf_2 = -(x/x_o)^{-1} a_1^2 \left[f_o''(\eta) \int_0^\eta \frac{\eta' - \beta}{f_o''(\eta')} d\eta' \right]^2 + 0(x^{-3/2} \ell nx)$$

A new variable $F_2(x,\eta)$ will now be introduced to achieve the homogeneous boundary condition at infinity and preserve the same two conditions at $\eta = 0$. With $F_2(x,\eta) = f_2 - \frac{1}{2} a_1^2 f_o(\eta) \left\{ \sum_{n=1,2}^{\infty} n (x/x_o)^{-n/2} \right\}$, the equation for F_2 is

$$MF_2 = -(x/x_o)^{-1} a_1^2 H_2(\eta) + 0\left\{ (x/x_o)^{-3/2} \ell n(x/x_o) \right\}$$

where

$$H_2(\eta) = \left[f_o''(\eta) \int_0^\eta \frac{\eta' - \beta}{f_o''(\eta')} d\eta' \right]^2 - (f_o')^2 + \tfrac{1}{2} f_o'' f_o$$

The leading term for F_2 is the $x^{-1}\, \ell \eta\, x$ term and is again obtained directly from the Green's function. The result for f_2 is

$$f_2(x,\eta) = a_1^2 \int_0^\infty f_o(\eta')H_2(\eta')d\eta' \ (x/x_o)^{-1} \ \ell n(x/x_o) \ (\eta f_o' - f_o)$$
$$+ 0 \ (x/x_o)^{-1} = -1.313 \ a_1^2 (x/x_o)^{-1} \ \ell n \ (x/x_o) \ (\eta f_o' - f_o)$$
$$+ 0 \ (x/x_o)^{-1}$$

For given ϵ, the asymptotic solution for large x is

$$\psi = (2xU\nu)^{\frac{1}{2}} \left\{ f_o(\eta) - ax^{-\frac{1}{2}} \int_0^\eta d\bar{\eta} \ f_o''(\bar{\eta}) \int_0^\eta \frac{\eta' - \beta}{f_o''(\eta')} d\eta' \right. \tag{3.19}$$
$$+ (1.70b - 1.313 \ a^2) \ x^{-1} \ \ell\eta \ x \ (\eta f_o' - f_o)$$
$$\left. - 0 \ (x^{-1}) \right\}$$

$\epsilon a_1 x_o^{\frac{1}{2}}$ and $\epsilon a_2 x_o$ are indentified as a and b such that the pressure variation agrees with the form assumed by Stewartson[1]. Eq. (3.19) is in agreement with that of Stewartson with the exception of the sign of the $\ell n \ x$ term due to a slip in [1].

3.3 Deviation of Initial Profile from Blasius Solution

When the perturbation parameter ϵ is a measure of the deviation of $f_o'(\eta)$ from the stream function, the initial profile at station $x = x_o$ can be written as

$$\psi(x,\eta) = (2x\nu U)^{\frac{1}{2}} \left[f_o(\eta) + \epsilon f_1(x,\eta) + \epsilon^2 f_2(x,\eta) + \dots \right]$$

The equation for $f_1(x,\eta)$ is $Mf_1 = 0$ where M is the linear operator defined in eq. (3.10a). The solution can be written in the form:

$$f_1(x,\eta) = \sum_{k=1,2}^{\infty} A_k (x/x_o)^{-\lambda_k/2} N_k(\eta)$$

The higher order solutions obey the nonhomogeneous equation

$$Mf_n(x,\eta) = H_n(x,\eta)$$

where $H_n(x,\eta)$ involves products of f_α and f_β with $\alpha + \beta = n$ for $n = 2, 3, \dots$ The lowest negative half power of x of terms in $H_n(x,\eta)$ is 4 and is greater that the only integer eigenvalue λ_1. From the general properties of the eigenvalue λ_k stated at the end of *3.1*, it is clear that a ℓnx term will never appear in any of the higher solutions $f_n(x,\eta)$. This is in contrast to the far wake solution in *2* where the eigenvalue are odd integers and ℓnx terms appear in the third and higher order perturbation solutions and are attached to all the eigensolutions except the first one.

3.4 Perturbations to Similar Solutions with Pressure Gradient

For a nonsimilar laminar boundary layer, the equation for the modified stream function $f(s,\eta)$ is

$$f_{\eta\eta\eta} + ff_{\eta\eta} + \beta(s) \ (1 - f_\eta^2) = 2s(f_\eta f_{\eta s} - f_s f_{\eta\eta})$$

where s and η are the variables defined in [10] and $\beta(s)$ is the pressure gradient parameter. For each constant β_o there is a similar solution, the Falkner-Skan solution $F_{\beta o}(\eta)$. The solution $f(s,\eta)$ is assumed to differ slightly from $F_{\beta o}(\eta)$ by a power series in ϵ, i.e.

$$f(s,\eta) = F_{\beta o}(\eta) + \epsilon f_1(s,\eta) + \epsilon^2 f_2(s,\eta) + \dots$$

The parameter ϵ can represent the small variation in initial profile or a measure of the variation in pressure gradient, $\beta(s) - \beta_o$. The homogeneous perturbation equation is of the form [22],

$$M_\beta f_n = 0$$

where

$$M_\beta = \frac{\partial^3}{\partial\eta^3} + F_{\beta o}\frac{\partial^2}{\partial\eta^2} - 2\beta F_{\beta o}'\frac{\partial}{\partial\eta} + F_{\beta o}'' + 2sF_{\beta o}''\frac{\partial}{\partial s} - 2sF_{\beta o}'\frac{\partial^2}{\partial s\partial\eta}$$

With the homogeneous boundary conditions the same as those for the case of zero pressure gradient, $\beta_o = 0$, stated in *3.1*, the eigenvalue problem can be set up in the same manner by looking for solution of the type $s^{-\lambda_k/2} N_k(\eta)$. The eigenvalues λ_k have been found by Chen and Libby [22] for several values of β_o between 2 and -0.1988, the separation value. With the exception of $\beta_o = 0$, none of the eigenvalues are integers and the sum of the eigenvalues is not an eigenvalue. It is clear that the perturbation solution due to the variations in the initial profile will not have any ℓns terms. The variation in pressure gradient can create ℓns terms in perturbation solutions only if the difference, $\beta(s) - \beta_o$ involves terms with special power of x, e.g. $x^{-\lambda_k/2}$

REFERENCES

1. Stewartson, K., On Asymptotic Expansions in the Theory of Boundary Layers, J.Math.Phys. 36, pp 173-191, 1957.
2. Goldstein, S., Flow of an Incompressible Viscous Fluid along a Semi-infinite Flat Plate, Tech.Rep.Eng.Res.Inst.Univ.Calif. No. HE-150-144, 1956.
3. Imai, I., Second Approximation to the Laminar Boundary Layer over a Flat Plate, J.Aero.Sc. 24, pp 155-156, February 1957.
4. Goldstein, S., Lectures on Fluid Mechanics, pp 136-144, John Wiley (Interscience), New York, 1960.
5. Van Dyke, M., Perturbation Methods in Fluid Mechanics, pp 139, Academic Press, New York, 1964.
6. Ting, L. and Chen, S., Perturbation Solutions and Asymptotic Solutions in Boundary Layer Theory, J.Engineering Math., 1, pp. 327-340, October 1967.
7. Goldstein, S., On the Two-Dimensional Steady Flow of a Viscous Fluid behind a Solid Body-I, Proc.Roy.Soc.A., 142, pp 545-563, 1933.
8. Schlichting, H., Boundary Layer Theory, McGraw-Hill Book Co., New York, 1955.
9. Courant, R. and Hilbert, D. Methods of Mathematical Physics vol.I, p 300, John Wiley (Interscience), 1953.

10. Libby, Paul A. and Fox, Herbert, Some Perturbation Solutions in Laminar Boundary Layer Theory, J. Fluid Mechanics 17, part 3, pp 433-449, 1963.

11. Tollmein, W., Handbuck der Experimental Physik 4, Part 1, p 269, 1931.

12. Carslaw, H.S. and Jaeger, J.C., Conduction of Heat in Solids, Oxford University Press, London, 1959.

13. Magnus, W., Oberhettinger, F. and Soni, R.P., Formulas and Theorems for the Special Functions of Mathematical Physics, Springer-Verlag, New York, 1966.

14. Crane, L.J., A note on Stewartson's Paper "On Asymptotic Expansions in the Theory of Boundary Layers" J. Math and Phys. 39 pp 172-174, 1960.

15. Lighthill, M.J., A Technique for Rendering Approximate Solutions to Physical Problems Uniformly Valid, Philos. Mag. [7], 40, pp 1179-1201, 1949.

16. Alden, H.L., Second Approximation to the Laminar Boundary Layer Flow over a Flat Plate, J. Math and Phys. 27, pp 91-104, 1948.

17. Friedrichs, K.O., Special Topics in Fluid Dynamics pp 120-130 New York University 1953.

18. Friedrichs, K.O., Special Topics in Analysis, New York University 1954.

19. Ting, L., On the Mixing of Two Parallel Streams, J. Math and Phys. 38, 3, pp 153-165, October 1959.

20. Libby, Paul A., Eigenvalues and Norms Arising in Perturbations about the Blasius Solution, AIAA Journal 3, 11, pp 2164-2165, November 1965.

21. Carrier, G.F. and Lin, C.C., On the Nature of the Boundary Layer Near the Leading Edge of a Flat Plate, Quart. Appl. Math. 6, pp 63-68, 1948.

22. Chen, K.K. and Libby, Paul A., On the Flows Nearly Described by the Falkner-Skan Equation, (Private Communication 1967) to be published in J. Fluid. Mech.

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